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# Equations of Motion of an Electrically Charged Particle in an Electromagnetic Field

# Equações de Movimento de uma Partícula Eletricamente Carregada em um Campo Eletromagnético

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**Abstract**

This article is intended for undergraduate students in exact sciences. The objective of this work is to determine in a didactic and detailed way the equations of motion  and  of an electrically charged particle that moves in a region with an electromagnetic field, with given initial conditions, considering the electric field  in the  plane and the magnetic field  in the direction of the  axis. Using the Lorentz force, second-order differential equations for velocity and position are obtained, which are solved by integration methods.
**Keywords:** Electric Field. Magnetic Field. Equations of Motion. Electromagnetic Field. Cycloid. Helical Trajectory.

**Resumo**

*Este artigo é destinado a estudantes de graduação em ciências exatas. O objetivo desse trabalho é determinar de forma didática e bem detalhada as equações de movimento  e  de uma partícula eletricamente carregada que se move em uma região com campo eletromagnético, com condições iniciais dadas, considerando o campo elétrico  no plano  e o campo magnético  na direção do eixo . Utilizando a força de Lorentz são obtidas equações diferenciais de segunda ordem para a velocidade e posição, que são solucionadas por métodos de integração.*
**Palavras-chave:** *Campo Elétrico. Campo Magnético. Equações de Movimento. Campo Eletromagnético. Cicloide. Trajetória Helicoidal.*

## 1 Introduction

This article addresses the motion of an electrically charged particle within a region of space containing both electric and magnetic fields. Key insights into the behavior of this particle are derived through classical treatment.

The particle's trajectory together with the force to which it is subjected depend on how this charged particle enters this region and the configuration of the electric and magnetic fields (REITZ, 1960). The objective of this work is to determine the equations of motion  and  of an electrically charged particle, with given initial conditions, considering the electric field  in the  plane and the magnetic field  in the direction of the  axis. Using the Lorentz force, second-order differential equations for velocity and position are obtained, which are solved by integration methods.

In Section 2 the theoretical method used in this work is presented and in Section 3 the equations of motion for the particle are obtained. Section 4 is reserved for conclusions and final comments.

## 2 Development

### 2.1 Theoretical treatment

It is known that a particle with charge  in the presence of an electric field is subjected to an electric force  given by (Vanderlinde, 2004; Meyer, 1972):

$$\vec{F}_e = q\vec{E} \qquad 01$$

where the electric force will have the same direction and direction as the electric field for  and will have the same direction and opposite direction of  for .

When entering a region with the presence of a magnetic field , a particle with an electric charge  will be subject to a force called magnetic force , which will depend on the way in which depends on the manner in which the particle gains access to this region. The magnetic force to which a particle with an electric charge  is subjected is given by (Zangwill, 2012):

$$\vec{F}_B = q\vec{v} \times \vec{B} \qquad 02$$

The vector product  present in Eq. (2) tells us that the direction of  is perpendicular to the vectors  and . Therefore, in regions where fields  and  are present, the electrically charged particle is subjected to an electromagnetic force , also known as Lorentz force, which is a force resulting from the sum of the forces  and , that is (Jackson, 1998):

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}\,. \qquad 03$$

In the next section, we will apply Eq. (3) in a specific situation, which holds significant physical importance. To elaborate further, we'll delve into the practical implications of this equation and explore its relevance within a specific context.

### 2.2 Motion in electromagnetic field

We will deal here with the situation in which a particle of charge and mass is launched into a region where there is an electric and magnetic field given by:

$$\vec{B} = B\hat{k}, \tag{04}$$

and

$$\vec{E} = E_y\hat{j} + E_z\hat{k}, \tag{05}$$

and considering the initial speed of the particle to be , that is, the particle is launched in the plane. Solving the vector product of Eq. (3), we have:

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix},$$

$$\vec{v} \times \vec{B} = v_yB\hat{i} - v_xB\hat{j}.$$

Considering this last equation and Eq. (5), Eq. (3) becomes:

$$\vec{F} = q(E_y\hat{j} + E_z\hat{k}) + q(v_yB\hat{i} - v_xB\hat{j}),$$

$$m\vec{a} = qE_y\hat{j} + qE_z\hat{k} + qv_yB\hat{i} - qv_xB\hat{j}.$$

The components of acceleration are:

$$\vec{a} = \dot{v}_x\hat{i} + \dot{v}_y\hat{j} + \dot{v}_z\hat{k}, \tag{06}$$

where the dot over the letter indicates the derivative with respect to time , as follows:

$$m(\dot{v}_x\hat{i} + \dot{v}_y\hat{j} + \dot{v}_z\hat{k}) = qv_yB\hat{i} + qE_y\hat{j} - qv_xB\hat{j} + qE_z\hat{k},$$

$$m(\dot{v}_x\hat{i} + \dot{v}_y\hat{j} + \dot{v}_z\hat{k}) = qv_yB\hat{i} + qE_y\hat{j} - qv_xB\hat{j} + qE_z\hat{k},$$

Defining:

$$\omega = \frac{qB}{m} \tag{07}$$

Note that using the SI units, is charge (in coulombs), is magnetic field (in teslas), and is mass (in kilograms):

$$\left[\frac{qB}{m}\right] = \frac{[C]\cdot[T]}{kg}.$$

The dimensions for charge (C), magnetic field (T), and mass (Kg) are respectively: $[C] = A \cdot s$ , $[T] = N \cdot A^{-1} \cdot m^{-1}$ , and $[kg] = kg$ . Substituting these dimensions:

$$\left[\frac{qB}{m}\right] = \frac{A \cdot s \cdot N \cdot A^{-1} \cdot m^{-1}}{kg} = \frac{s \cdot N}{kg \cdot m}.$$

Simplifying, we get:

$$\left[\frac{qB}{m}\right] = s^{-1}.$$

Therefore, is equivalent to the dimension of frequency (). In SI units, it represents , which is a measure of angular frequency.

And from the vector equality in the previous equation to Eq. (7), we have:

$$\dot{v}_x = \omega v_y, \tag{08}$$

$$\dot{v}_y = \frac{qE_y}{m} - \omega v_x, \tag{09}$$

$$\dot{v}_z = \frac{qE_z}{m}. \tag{10}$$

Eq. (10) is easy to solve, since $a_z = qE_z/m$ is a constant. Thus, the equations governing the motion of the -component are those of uniformly accelerated motion, equations well-established in the study of kinematics (Santos, 2022). In fact, by Eq. (10):

$$\frac{dv_z}{dt} = \frac{qE_z}{m} \Rightarrow dv_z = \frac{qE_z}{m}dt,$$

$$\int_{v_{0z}}^{v_z} dv_z = \int_0^t \frac{qE_z}{m}dt,$$

$$v_z(t) = v_{0z} + \frac{qE_z}{m}t \tag{11}$$

The component of the position can be obtained by integrating Eq. (11):

$$\frac{dz}{dt} = v_{0z} + \frac{qE_z}{m},$$

$$dz = v_{0z}dt + \frac{qE_z}{m}tdt,$$

$$\int_{z_0}^{z} dz = \int_0^t v_{0z}\,dt + \frac{qE_z}{m}\int_0^t t\,dt,$$

$$z(t) - z_0 = v_{0z}t + \frac{qE_z}{m}t^2,$$

$$z(t) = z_0 + v_{0z}t + \frac{qE_z}{m}t^2. \tag{12}$$

Let us now determine the equations of motion and . Deriving Eq. (8) in relation to time, yields:

$$\ddot{v}_x = \omega\dot{v}_y. \tag{13}$$

Inserting Eq. (9) into Eq. (13), we obtain:

$$\ddot{v}_x + \omega^2 v_x = \omega\frac{qE_y}{m}. \tag{14}$$

This is an inhomogeneous differential equation, whose general solution is (for its detailed solution see Appendix A):

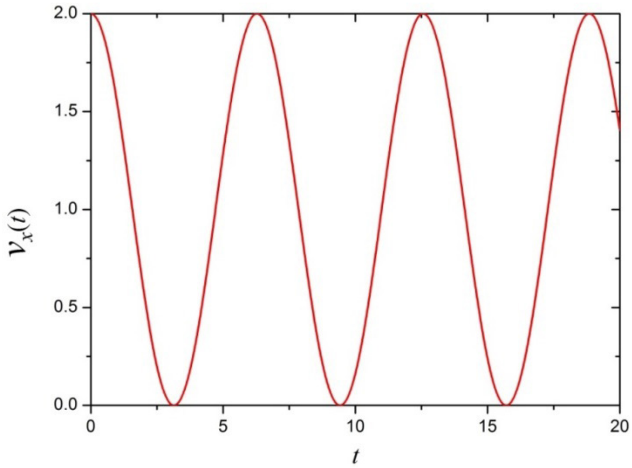
$$v_x(t) = c_1\cos(\omega t) + c_2\,\mathrm{sen}(\omega t) + \frac{E_y}{B}.$$

Just for simplicity in the following developments, we set, indicating a choice for the initial phase of the function. In this manner:

$$v_x(t) = c_1 \cos(\omega t) + \frac{E_y}{B}\,. \qquad 15$$

It can be seen from the expression above that the speed $v_x(t)$ is a periodic function with $v_{max} = E_y/B + c_1$ and $v_{min} = E_y/B - c_1$. Fig. 1 shows the graph of the function $v_x(t)$ for $c_1 = 1$, $\omega = 1$ and $E_y/B = 1$.

**Figure 1 -** Graph of the component of velocity as a function of time



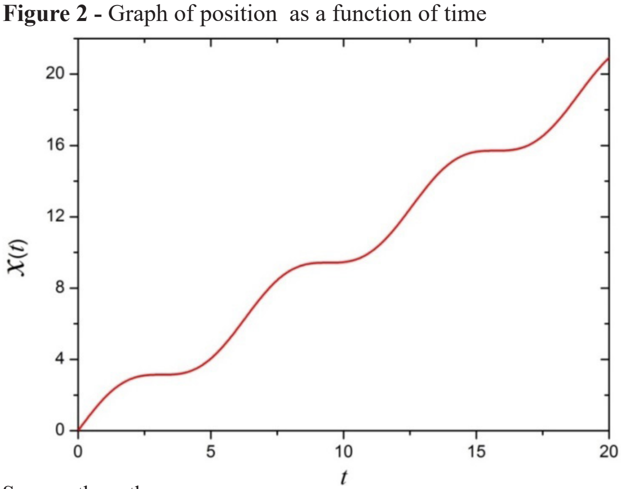

**Source:** the author.

To determine the component of the position, we integrate Eq. (15):

$$x(t) = x_0 + \frac{c_1}{\omega}\mathrm{sen}(\omega t) + \frac{E_y}{B}t. \qquad 16$$

Unlike the function , the function has no upper or lower limit. Fig. 2 shows the graph of the function for , , and .

**Figure 2 -** Graph of position as a function of time




**Source:** the author.

Let us now determine the component of the velocity. Inserting Eq. (15) into Eq. (9), we have:

$$\dot{v}_y = \frac{qE_y}{m} - \omega\left[c_1\cos(\omega t) + \frac{E_y}{B}\right],$$

$$\frac{dv_y}{dt} = \frac{qE_y}{m} - \omega c_1 \cos(\omega t) - \omega\frac{E_y}{B}\,,$$

remembering that , in the equation above the last term to the right of the equality cancels with the first term , yields:

$$dv_y = -\omega c_1 \cos(\omega t)\,dt,$$

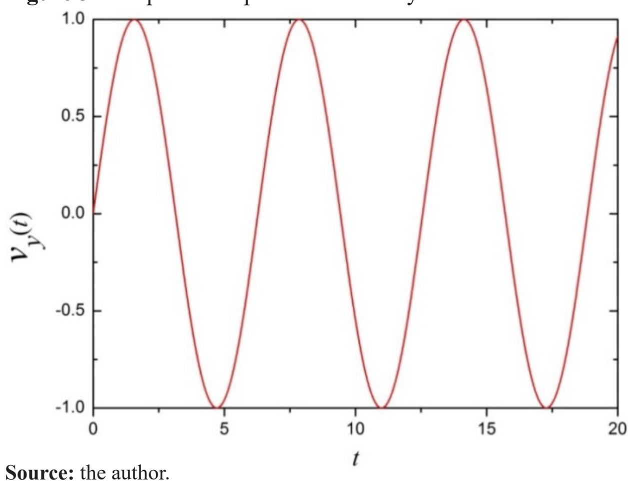
$$\int_0^{v_y} dv_y = -\omega c_1 \int_0^t \cos(\omega t)\,dt,$$

$$v_y(t) = \frac{\omega c_1}{\omega}[\mathrm{sen}(\omega t)]\Big|_0^t\,,$$

$$v_y(t) = c_1\mathrm{sen}(\omega t). \qquad 17$$

Figure 3 shows the graph of the function for and .

**Figure 3 -** Graph of component of velocity:




**Source:** the author.

To obtain the component of the position, we simply integrate Eq. (17):

$$\frac{dy}{dt} = c_1\mathrm{sen}(\omega t),$$

$$dy = c_1\mathrm{sen}(\omega t)dt,$$

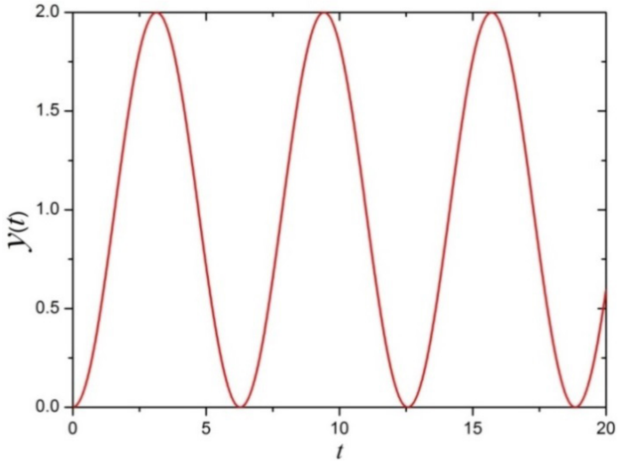
$$\int_{y_0}^{y} dy = c_1\int_0^t \mathrm{sen}(\omega t)\,dt,$$

$$y(t) - y_0 = -\frac{c_1}{\omega}[\cos(\omega t)]\Big|_0^t\,,$$

$$y(t) = y_0 + \frac{c_1}{\omega}[1 - \cos(\omega t)]. \qquad 18$$

It can be observed from the above expression that the component of the position is a periodic function constrained between a minimum and a maximum point. Figure 4 depicts the graph of the function for and .

**Figure 4 -** Graph of the  component of position as a function of time




**Source:** the author.

Summarizing, for an electrically charged particle with a charge  and a mass  entering an electromagnetic field given by Eqs. (4) and (5), and considering , we obtain the particle speed , with:

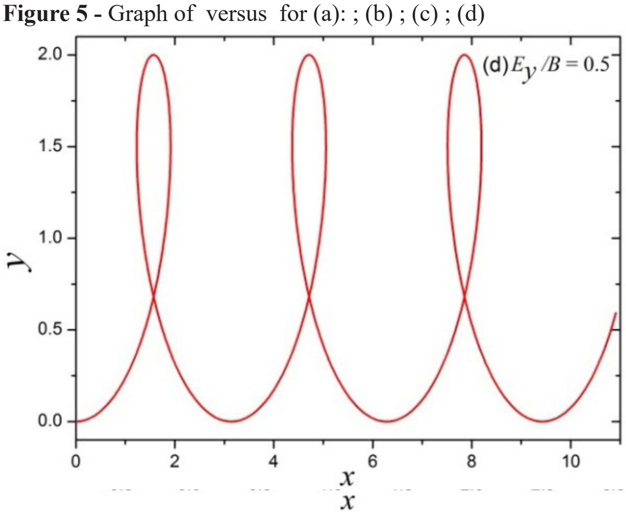
$$\begin{cases} v_x(t) = c_1\cos(\omega t) + \dfrac{E_y}{B}, & (15) \\ v_y(t) = c_1 \mathrm{sen}(\omega t), & (17) \\ v_z(t) = v_{0z} + \dfrac{qE_z}{m}t, & (11) \end{cases}$$

and the position of the particle , with:

$$\begin{cases} x(t) = x_0 + \dfrac{c_1}{\omega}\mathrm{sen}(\omega t) + \dfrac{E_y}{B}t, & (16) \\ y(t) = y_0 + \dfrac{c_1}{\omega}[1 - \cos(\omega t)], & (18) \\ z(t) = z_0 + v_{0z}t + \dfrac{qE_z}{m}t^2, & (12) \end{cases}$$

It is interesting to study the projection of movement in the  plane, whose shape will depend on the  ratio. Considering , , and , Fig. 5 shows some possibilities: Figure 5(a): ; Figure 5(b): ; Figure 5(c): ; Figure 5(d): .

**Figure 5 -** Graph of  versus  for (a): ; (b) ; (c) ; (d)




**Source:** the author.

For specific values of the constants , , and , the projection of the movement on the  plane is a cycloid (see Appendix B).

Another interesting specific scenario is if . In this case, Eqs. (18) and (16) become:

$$y(t) - (y_0 - c_1/\omega) = -\cos(\omega t),$$

$$y(t) - C_y = -\frac{c_1}{\omega}\cos(\omega t),$$

$$x(t) - x_0 = \frac{c_1}{\omega}\mathrm{sen}(\omega t),$$

$$x(t) - C_x = \frac{c_1}{\omega}\mathrm{sen}(\omega t),$$

which represent the equation of a circle (Leithold, 1982) with center at , with  and . Note that:

$$[x(t) - C_x]^2 + [y(t) - C_y]^2 = \left(\frac{c_1}{\omega}\right)^2\cos^2(\omega t) + \left(\frac{c_1}{\omega}\right)^2\mathrm{sen}^2(\omega t),$$

$$[x(t) - C_x]^2 + [y(t) - C_y]^2 = \left(\frac{c_1}{\omega}\right)^2,$$

that is, the radius  of the circumference is:

$$R = \frac{c_1}{\omega}. \tag{19}$$

The speed  in the  plane in this case of  is (see Eqs. (15) and (17)):

$$\vec{v}_{xy}(t) = v_x(t)\hat{\imath} + v_y(t)\hat{\jmath},$$

$$\vec{v}_{xy}(t) = c_1\cos(\omega t)\,\hat{\imath} + c_1\mathrm{sen}(\omega t)\hat{\jmath},$$

whose module is:

$$|\vec{v}_{xy}(t)| = \sqrt{v_x(t)^2 + v_y(t)^2},$$

$$|\vec{v}_{xy}(t)| = \sqrt{(c_1)^2\cos^2(\omega t) + (c_1)^2\mathrm{sen}^2(\omega t)},$$

$$|\vec{v}_{xy}(t)| = \sqrt{(c_1)^2[\cos^2(\omega t) + \mathrm{sen}^2(\omega t)]},$$

$$|\vec{v}_{xy}(t)| = \sqrt{(c_1)^2} = c_1 = \text{constant}.$$

That is,  is a constant that is the particle velocity modulus in the  plane. Using Eq. (19) we have:

$$|\vec{v}_{xy}(t)| = c_1 = \omega R. \tag{20}$$

Note also that the acceleration in the  plane with  is:

$$\vec{a}_{xy}(t) = \dot{v}_x(t)\hat{\imath} + \dot{v}_y(t)\hat{\jmath},$$

$$\vec{a}_{xy}(t) = -\omega c_1\,\mathrm{sen}(\omega t)\,\hat{\imath} + \omega c_1\cos(\omega t)\hat{\jmath},$$

whose absolute value is:

$$|\vec{a}_{xy}(t)| = \sqrt{a_x(t)^2 + v_y(t)^2},$$

$$|\vec{a}_{xy}(t)| = \sqrt{(\omega c_1)^2\mathrm{sen}^2(\omega t) + (\omega c_1)^2\cos^2(\omega t)},$$

$$|\vec{a}_{xy}(t)| = \sqrt{(\omega c_1)^2[\mathrm{sen}^2(\omega t) + \cos^2(\omega t)]},$$

$$|\vec{a}_{xy}(t)| = \sqrt{(\omega c_1)^2} = \omega c_1,$$

$$|\vec{a}_{xy}(t)| = \omega|\vec{v}_{xy}(t)| = \text{constant}.$$

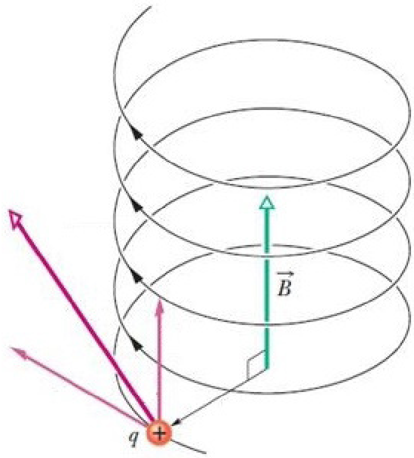
By Eq. (20) we have that , and the previous equation is:

$$|\vec{a}_{xy}(t)| = \frac{|\vec{v}_{xy}(t)|^2}{R}, \qquad 21$$

with being the well-known centripetal acceleration (SERWAY; JEWETT, 2018).

The velocity component , given by Eq. (11), parallel to the magnetic field causes the particle to move in the same direction as the magnetic field. In this scenario, where , the particle trajectory is helical with the axis in the same direction as the magnetic field , as illustrated in Figure 6.

**Figure 6** - The particle describes a helical trajectory of radius .




**Source:** Duarte (2024).

## 3 Conclusion

The general objective of this work was to determine the equations of motion of a particle with an electric charge and mass moving in an electromagnetic field. To this end, a quick theoretical review was made of the electrical and magnetic forces that act on a charged particle due to the action of these fields. The components of velocity () and position () were then determined and analyzed. The particular case was adopted in which the electric field is in the plane () and the magnetic field in the direction of the axis (), considering the initial speed of the particle , that is, the particle is launched in the plane. By applying the Lorentz force, second-order differential equations are derived and subsequently solved using integration methods. The shape of the motion curve projected on the plane depends on the ratio. The velocity component , given by Eq. (11), parallel to the magnetic field causes the particle to move in the same direction as the magnetic field, and in the situation where the particle trajectory it is helical with the axis in the same direction as the magnetic field . Despite having been applied to a particular case of electric () and magnetic () fields, the method used here can be applied to several other situations of charged particles moving in electromagnetic fields. The book of João Barcelos Neto (NETO, 2004) determines the same equations obtained in this article, however in a different way.

**Appendix A**

Eq. (14) is a non-homogeneous second-order differential equation with constant coefficients, of the type (RODRIGUES, 2017)

$$a\ddot{\xi}(t) + b\dot{\xi}(t) + c\xi(t) = \delta(t), \qquad (A1)$$

whose solution is

$$\xi(t) = \xi_h(t) + \xi_p(t), \qquad (A2)$$

where $\xi_h(t)$ is the solution of the homogeneous differential equation and $\xi_p(t)$ is a particular solution of Eq. (A1). The solution of the homogeneous equation is:

$$\xi_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}\ (\text{if } r_1 \neq r_2), \qquad (A3a)$$

and

$$\xi_h(t) = C_1 e^{rt} + C_2 t e^{rt}\ (\text{if } r_1 = r_2), \qquad (A3b)$$

where $r_1$ and $r_2$ can be determined by the indicial equation:

$$ar^2 + br + c = 0. \qquad (A4)$$

Comparing the coefficients of Eq. (A1) with Eq. (14), the indicial equation (A4) takes the form:

$$1 \cdot r^2 + 0 \cdot r + \omega^2 = 0,$$

$$r^2 = -\omega^2 \Rightarrow r = \mp\sqrt{\omega^2} = \pm i,$$

and

$$r_1 = i\omega \text{ and } r_2 = -i.$$

Inserting $r_1$ and $r_2$ into Eq. (A3a) we have:

$$\xi_h(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}.$$

Using Euler's formula (BUTKOV, 1973):

$$\xi_h(t) = C_1[\cos(\omega t) + i\,\mathrm{sen}(\omega t)] + C_2[\cos(\omega t) - i\,\mathrm{sen}(\omega t)],$$

$$\xi_h(t) = (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\mathrm{sen}(\omega t),$$

defining $c_1 = C_1 + C_2$ e $c_2 = i(C_1 - C_2)$:

$$\xi_h(t) = c_1\cos(\omega t) + c_2\,\mathrm{sen}(\omega t). \qquad (A5)$$

Comparing Eq. (A1) with Eq. (14) it is noted that: $\delta(t)$ is a constant, that is, $\delta(t) = \delta = \omega q E_y/m$. Thus, after substituting $\omega = qB/m$, it can be verified by inspection that the particular solution $\xi_p(t) = E_y/B$. Inserting $\xi_h(t)$ and $\xi_p(t)$ in Eq. (A2) we have:

$$\xi(t) = c_1\cos(\omega t) + c_2\,\mathrm{sen}(\omega t) + \frac{E_y}{B}. \qquad (A5)$$

**Appendix B**

A cycloid is the curve traced by a point on a circle as it rolls along a straight line without slipping (FLORIAN, 1999). The parametric equations for a cycloid are:

$$x = \mathcal{R}(\theta - \sin\theta), \qquad (B1)$$

and

$$y = \mathcal{R}(1 - \cos\theta). \qquad (B2)$$

Let Eq. (16) be with $x_0 = 0$:

$$x(t) = \frac{E_y}{B}t + \frac{c_1}{\omega}\mathrm{sen}(\omega t),$$

$$x(t) = \frac{E_y}{B\omega}\omega t + \frac{c_1}{\omega}\mathrm{sen}(\omega t).$$

Considering the specific case in which $E_y/B = -c_1$:

$$x(t) = -\frac{c_1}{\omega}\omega t + \frac{c_1}{\omega}\mathrm{sen}(\omega t).$$

Defining: $\mathcal{R} = c_1/\omega$ and $\theta = -\omega t$:

$$x(t) = \mathcal{R}\theta + \mathcal{R}\,\mathrm{sen}(-\theta) = \mathcal{R}\theta - \mathcal{R}\,\mathrm{sen}(\theta),$$

$$x(t) = \mathcal{R}[\theta - \mathrm{sen}\,\theta],$$

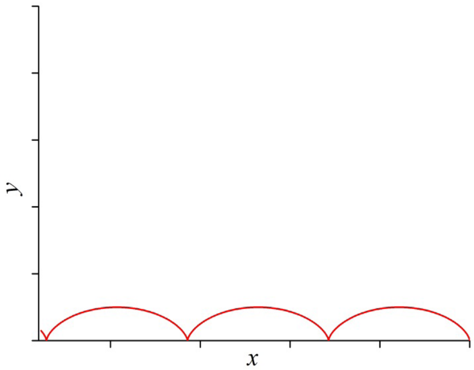
which is equal to Eq. (B1). For $y_0 = 0$, Eq. (18) becomes:

$$y(t) = \mathcal{R}[1 - \cos(-\theta)],$$

$$y(t) = \mathcal{R}[1 - \cos(\theta)],$$

which is equal to Eq. (B2). Therefore, for the specific case in which $E_y/B = -c_1$, the projection of the movement on the XY plane is a cycloid. Fig. 7 shows y versus x for $c_1 = 1$, $\omega = 1$, and $E_y/B = -1$.

**Figure 7 -** Graph of  versus  (a cycloid).

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